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Brittle detachment of a stiffener bonded to an elastic plate

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Abstract. The diffusion of contact stresses between an elastic bar, bonded to an elastic half-plane and loaded longitudinally, requires the integration of a singular integral equation. The solution of this equation is not available in closed form, but only by a series expansion of the contact tangential force mutually transmitted between the stiffener and the plate. Since forty years it has been realized that the expansion of the solution in terms of Chebyshev polynomials is its most convenient method of representation. The procedure can also be extended to treat the brittle detachments of the tips of the stiffener when, according to Griffith's criterion of fracture, a balance can be virtually established between the increase of strain energy due to a propagation of cracks and the surface energy created.

Key words: brittle fracture, Chebyshev polynomials, elastic stiffeners.

1. Introduction

The diffusion of stresses between an elastic one-dimensional filament and an elastic half-plane to which the filament is bonded is a classic mixed-boundary-value problem of linear elasticity tackled with different methods since the time when it was proposed (but not solved) by E. Reissner [1]. The difficulty is due to the fact that the problem involves an integro-differential equation defined on a finite interval, and the solution of this equation can only be approximated either by series expansions, by successive approximations, or by collocation.

However, among these methods, the solution in terms of Chebyshev polynomials proved to be the most elegant and efficient one. The series representing the solution is rapidly convergent, and, more importantly, the singularities of the solution at the tips of the reinforcement are explicitly indicated and completely contained in a singular function which multiplies the whole series. This implies that the nature of the singularity is always the same, independent of the number of terms with which the series is approximated. The use of Chebyshev polynomials in a singular integro-differential equation, like that of the finite reinforcement considered here, has *de facto* been introduced by Hamel [2, p. 145] in his book on integral equations. Hamel used combinations of trigonometric functions without discussing the completeness of the system of these functions. Completeness is, however, well established for Chebyshev functions (*cf.* Weinberger [3, p. 73]). The application of Chebyshev functions in solving the problem of the finite stiffener appears much later in the literature by merit of Arutiunian [4] and Bufler [5], who worked independently. A clear explanation of the procedure is now recorded in the book by Grigolyuk and Tolkachev [6, p. 176].

There is an interesting extension of the problem of the stiffener, assuming that the stiffener is welded to the half-plane by a brittle thin sheet. Consider the case in which two concentrated forces of equal magnitude but opposite directions are applied to its ends. If these forces are sufficiently small, the transmission of stresses between stiffener and half-plane is determined

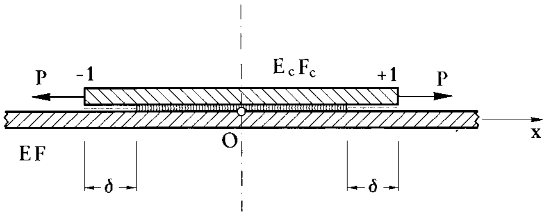

Figure 1. Geometrical sketch.

by solving (with one of the approximation procedures at hand) Hamel's equation. But, if loads exceed a certain limit, it may happen that a partial detachment of the stiffener occurs starting from the ends, where stresses become theoretically infinite. The critical value of the magnitude of loads at which the propagation of detachment initiates can easily be determined by a balance between the energy released by the detachment and that supplied by the debonding of the welding material. This is, conceptually, a simple variant of Griffith's fracture criterion.

But the problem of evaluation of the critical load of possible debonding of a stiffener can be further extended to the case in which it is loaded by impressing two equal and opposite displacements at its end instead of two opposite forces. The determination of the stress diffusion between stiffener and plate under prescribed displacements at the ends of the stiffener requires the solution of an integral equation, again treatable with Chebyshev polynomials. The surprising result is that the onset of rupture by impressed displacements is, not only numerically, but also qualitatively different from that of impressed forces. This fact has also been observed by Burridge and Keller [7] in another problem.

There are many sectors of technology where the brittle detachment of a stiffener from a plate arises as the simplest model for describing more complex situations. The rupture of a weld in a steel beam, or the partial detachment of a rib from a panel represent the most popular instances. But it has recently been realized that the same problem must be solved in seismology in order to determine the limiting condition at which a fault, bonded to a substrate, suddenly detaches and eventually slips, as Knopoff *et al.* [8] have demonstrated.

2. The two-bars model

In order to illustrate the essential features of the problem under consideration we consider a system of two bonded bars, one of length 2 and extensional rigidity $E_c F_c$, and one of infinite length and extensional rigidity EF (Figure 1).

The two bars are welded together by an adhesive substance which is rigid at low level of load, but breaks when its energy for unit length reaches a given value γ. We assume that only the upper bar is loaded at its ends by two equal and opposite forces P, and only a tangential interaction force occurs between the two bars. Let us denote the line of adhesion by x and take the origin at the mid-point of the upper bar so that it covers the interval $-1 \le x \le 1$.

If the bars are sufficiently slender, we can apply the elementary theory to them. This implies that it is not necessary to calculate the tangential forces along the line of contact, but only the axial forces N_c and N absorbed, respectively, by the stiffener and the part of the other bar attached to it. The result is

$$N_c = P\frac{E_c F_c}{EF + E_c F_c}, \quad N = P\frac{EF}{EF + E_c F_c}. \tag{2.1}$$

The part $|x| > 1$ of the infinite bar exterior to the interval $|x| \leq 1$ is free from any force.

Then the total elastic energy of the system, that is, the sum of the strain energy minus the external work of the loads, is given by

$$E_{\mathrm{tot}} = -\frac{1}{2}\frac{2P^2}{EF + E_c F_c}, \tag{2.2}$$

where 2 is the length of the stiffener.

Let us now suppose that the connection between the bars is not complete as two cracks of extent δ are localized at the tips of the interval $|x| = 1$ (Figure 1). In this case the axial forces are not as before but have the values

$$\begin{cases} N_c = P & \text{for} \quad |1 - x| < \delta, \\ N_c = P\dfrac{E_c F_c}{EF + E_c F_c}, \quad N = P\dfrac{EF}{EF + E_c F_c} & \text{for} \quad |x| < 1 - \delta, \\ N = 0 & \text{for} \quad |x| > 1 - \delta. \end{cases} \tag{2.3a,b,c}$$

On the other hand, according to Griffith's criterion for fracture, the creation of two symmetric cracks of lengths δ requires a surface energy $2\gamma\delta$. Thus, the total energy of the partially fractured system becomes

$$E_{\mathrm{tot}} = -\frac{P^2\delta}{E_c F_c} - \frac{P^2(1-\delta)}{E_c F_c + EF} + 2\gamma\delta. \tag{2.4}$$

The propagation of the cracks occurs when P reaches a value such that E_{tot} is stationary with respect to δ. But E_{tot} is linear in δ ad hence stationarity of E_{tot} requires the coefficient of δ to vanish. Thus from (2.4) we obtain the critical value of P for which cracks advance provoking, eventually, the complete detachment:

$$P_{\mathrm{cr}}^2 = \frac{2\gamma E_c F_c}{1 - \dfrac{1}{1 + \frac{Ef}{E_c F_c}}}. \tag{2.5}$$

The analysis of the case in which two equal and opposite horizontal displacements are impressed at the end of the stiffener is again based on inspection of the total energy. Let us denote the common magnitude of these displacements by U. If U is sufficiently small, so that the welding between the two bars is complete, the forces at the tips associated to the displacements U are

$$P = \frac{U}{E_c F_c + EF}, \tag{2.6}$$

and the corresponding total energy is

$$E_{\mathrm{tot}} = \frac{1}{2}\frac{2U^2}{(E_c F_c + EF)}. \tag{2.7}$$

In the presence of two symmetric cracks of lengths δ situated at the extremities of the stiffer (see again Figure 1) the relationship between the forces P and the displacements U becomes

$$U = P\left(\frac{\delta}{E_c F_c} + \frac{1-\delta}{E_c F_c + EF}\right). \tag{2.8}$$

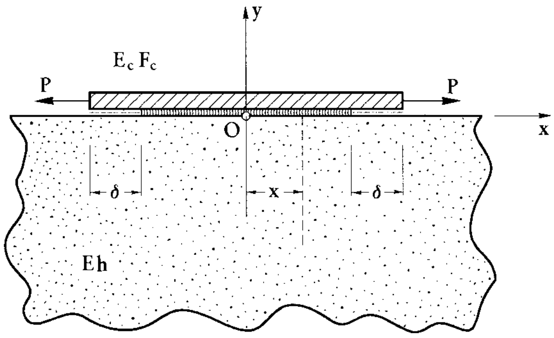

Figure 2. Stiffener partially attached to elastic half-plane.

Considering that here again the fracture energy is $2\gamma\delta$, we find that the total energy of the system has the expression

$$E_{\mathrm{tot}} = \frac{1}{2}\frac{2U^2}{\dfrac{\delta}{E_c F_c} + \dfrac{1-\delta}{E_c F_c + EF}} + 2\gamma\delta. \tag{2.9}$$

Now the energy becomes stationary with respect to δ as soon as U^2 reaches the value

$$U_{\mathrm{cr}}^2 = \frac{2\gamma E_c F_c (E_c F_c + EF)}{(E_c F_c + \delta EF)}. \tag{2.10}$$

Contrary to the case in which two forces are impressed, we now find that U_{cr}^2 does not depend only on γ and the extensional rigidities of the bars, but also on the length δ of the cracks.

3. Load transfer from a finite stiffener and a half-plane

The diffusion of stresses between an elastic stiffener bonded to an elastic semi-infinite plate offers conceptually the same problem, but now the evaluation of the interaction force between the stiffener and the plate is much more difficult. The situation we must describe is that of a stiffener of extensional rigidity $E_c F_c$ partially attached to an elastic half-plane of modulus E and thickness h (Figure 2).

Let the stiffener occupy the interval $-1 \le x \le 1$ of the x-axis of a system of Cartesian x, y-axes chosen as shown in Figure 2, so that the middle plane of the plate occupies the lower half-plane $y < 0$. The stiffener is attached to the plate along the interior part $|x| < 1-\delta$, while the end intervals $1-\delta < |x| < 1$ are detached. Two longitudinal forces P are applied to the ends of the stiffener. We assume that the stiffener transmits only tensile or compressive force, therefore only a tangential stress $q(x)$ is transmitted between the stiffener and the plate.

Consider the equilibrium of the part of the stiffener confined between a section of abscissa x and the end $(1-\delta)$ (Figure 2). The axial force at x is thus

$$N_c = P - \int_x^{1-\delta} q(x)\,\mathrm{d}x, \tag{3.1}$$

and the corresponding strain ϵ_c is $N_c/E_c F_c$. Considering instead the plate, the strain ϵ_x along the line of action of the forces $q(x)$ is given by the formula of plane elasticity (*cf.* Szabó [9, Chapter 11, p. 212])

$$\epsilon_x = \frac{2}{\pi Eh} \int_{-(1-\delta)}^{1-\delta} \frac{q(x)}{x - x_0} \mathrm{d}x. \tag{3.2}$$

By equating the two strains we obtain the integral equation

$$\int_{-(1-\delta)}^{1-\delta} \frac{q(x)}{x - x_0} \mathrm{d}x = \frac{\pi^4 \lambda}{4} \left(P - \int_{x_0}^{1-\delta} q(x) \mathrm{d}x \right), \tag{3.3}$$

where (*cf.* Grigolyuk-Tolkachev [6, Section 3.7]) we have put $\lambda = \frac{2}{\pi} \frac{Eh}{E_c F_c}$ The solution of (3.3) is also subject to the quilibrium condition

$$\int_{-(1-\delta)}^{1-\delta} q(x) \mathrm{d}x = 0. \tag{3.4}$$

In order to treat (3.3) by using Chebyshev's polynomials we make the change of variable (*cf.* Grigolyuk-Tolkachev [6, Section 3.8])

$$t = \frac{1}{1-\delta} x$$

and transform (3.3), (3.4) into

$$\int_{-1}^{1} \frac{q(t)}{t - t_0} \mathrm{d}t = \frac{\pi^2 \lambda}{4} \left(P - (1-\delta) \int_{t_0}^{1} q(t) \mathrm{d}t \right), \tag{3.5}$$

$$\int_{-1}^{1} q(t) \mathrm{d}t = 0. \tag{3.6}$$

Let us now seek an expansion of the solution $q(x)$ in terms of Chebyshev's polynomials of the first kind $T_s(t)$

$$q(t) = \frac{P}{\pi} \frac{1}{\sqrt{1-t^2}} \sum_{s=1,3,\ldots}^{2n+1} X_s T_x(t), \tag{3.7}$$

where X_s are constants to be determined. Note that (3.7) contains only odd terms since $q(t)$ must be an odd function of t in the interval $-1 \le t \le 1$. As a consequence, condition (3.6) is automatically satisfied. We now substitute (3.7) in (3.5) and compute the integrals, while still using the properties of Chebyshev functions. The result is the equation

$$\sum_{s=1,3,\ldots}^{2n+1} X_s U_{s-1}(t_0) + \frac{\pi}{4} \lambda (1-\delta) \sum_{s=1,3,\ldots}^{2n+1} \frac{X_s}{s} U_{s-1}(t_0) \sqrt{1-t_0^2} = \frac{\lambda \pi^2}{4}, \tag{3.8}$$

which is structurally similar, but not identical, to that recorded in the book by Grigolyuk-Tolkachev [6, Section 3.8] in the treatment of a non-detached stiffener. Equation (3.8) can be solved by Bubnov-Galerkin's method, which consists in multiplying both sides of the equation by $\sqrt{1-t_0^2 U_{j-1}(t_0)}(j = 1, 3, \ldots, 2n+1)$ and integrating over the interval $-1 \le t \le 1$, still making recourse to the definite integrals of Chebyshev functions (*cf.* [10]). As result we obtain the set of algebraic equations for X_j

$$X_j + \frac{\lambda}{2}(1-\delta) \sum_{s=1,3,\ldots}^{2n+1} a_{js} X_s = \frac{\lambda}{2} b_j \quad (j = 1, 3, \ldots, 2n+1), \tag{3.9}$$

where the coefficients a_{js}, already calculated by Grigolyuk and Tolkachev [6, Section 3.8], have the form

$$a_{js} = -\frac{4j}{[(j+s)^2-1][(j-s)^2-1]} \quad (j, s = 1, 3, \dots). \tag{3.10}$$

The terms b_j have the values

$$b_1 = \frac{\pi^2}{2}, \quad b_3 = b_5 = \dots = 0. \tag{3.11}$$

Solving the system of algebraic equations (3.9), we determine the X_j and hence $q(t)$. Then from (3.2), written after introduction of the t-variable, we obtain the axial strain

$$\epsilon_x = \epsilon_c = \frac{2}{\pi E h} \int_{-1}^{1} \frac{q(t)}{t - t_0} \mathrm{d}t = \frac{2}{\pi E h} P \sum_{s=1,3,\dots}^{2n+1} X_s U_{s-1}(t), \tag{3.12}$$

and hence, by integration with respect to x expressed in terms of t, we find the axial displacement in the stiffener:

$$u(t_0) = \frac{2}{\pi E h} \int_0^{t_0} (1-\delta) \sum_{s=1,3,\dots}^{2n+1} X_s U_{s-1}(t_0) \mathrm{d}t_0 = \frac{2}{\pi E h} P(1-\delta) \sum_{s=1,3,\dots}^{2n+1} \frac{X_s}{s} T_s(t_0). \tag{3.13}$$

In particular, the displacements of the points $t_0 = \pm 1$, are

$$u(\pm 1) = \pm \frac{2}{\pi E h} P(1-\delta) \sum_{s=1,3,\dots}^{2n+1} \frac{X_s}{s}. \tag{3.14}$$

The only exterior forces acting on the system are the two opposite forces P applied at the ends of the stiffener (Figure 2). Thus, the total energy of the partially cracked system is

$$E_{\mathrm{tot}} = -\frac{1}{2} 2 \frac{2P^2}{\pi E h} (1-\delta) \sum_{s=1,3,\dots}^{2n+1} \frac{X_s}{s} - \frac{1}{2} 2 \frac{P^2 \delta}{E_c F_c} + 2\gamma\delta, \tag{3.15}$$

where the first term represents the strain energy in the piece of stiffener bonded to the plate plus that stored in the plate; the second term is the strain energy of the cracked portions of stiffener and the third is the fracture energy.

The discussion of the critical condition of (3.15) is not easy, because the X_s also depend on δ as we solve system (3.9). However, some useful indications about the critical values of P can be obtained by examining two limiting cases. Consider first the case in which λ, defined by the relation $\lambda = \frac{2}{\pi} \frac{E h}{E_c F_c}$, is very small with respect to 1. Then an approximate solution of system (3.9) is given by the following values of the X_s

$$X_1 \simeq \lambda \frac{\pi^2}{4}, \quad X_3 \simeq X_5 \simeq \dots \simeq 0. \tag{3.16}$$

If we replace this expression of the X_s in (3.15) and impose the condition of stationarity of ϵ_c with respect to δ, we obtain

$$P_{\mathrm{cr}}^2 = \frac{2\gamma E_c F_c}{1 - \dfrac{2E_c F_c}{E h} \lambda \dfrac{\pi}{4}}. \tag{3.17}$$

Recalling that $\lambda = \frac{2}{\pi}\frac{E\,h}{E_c F_c}$ we thus conclude that P_{cr}^2 is theoretically infinite, which result is not surprising because, if the plate is very soft with respect to the bar, it behaves as if it were detached. Let us instead suppose that the product $(1-\delta)\lambda$ is large. This implies that not only λ must be large, but also δ sufficiently close to zero. In this case an approximate solution of system (3.9), obtained by neglecting the first terms X_j in (3.9) and exploiting the fact that coefficients of the form a_{jj} are dominant, is

$$X_1 \simeq \frac{3\pi^2}{8(1-\delta)}, \quad X_3 \simeq X_5 \simeq \ldots \simeq 0. \tag{3.18}$$

Substitution of this value of X_1 in (3.15) yields the following expression for the total energy

$$E_{\text{tot}} = -\frac{1}{2}2\frac{P^2}{\pi\,E\,h}\frac{3\pi^2}{8} - \frac{1}{2}2\frac{P^2\delta}{E_c F_c} + 2\gamma\delta. \tag{3.19}$$

The energy becomes stationary for

$$P_{\text{cr}}^2 = 2\gamma E_c F_c. \tag{3.20}$$

This is exactly the critical value of the two forces which produce the brittle detachment of an elastic stiffener from a rigid support.

4. Detachment due to impressed displacements

Figure 2 illustrates the case in which the stiffener is loaded by two opposite forces P. But let us suppose that, instead of forces, the stiffener is loaded by two equal displacements U at its ends, and we want to determine the critical value of U for which two pre-existing cracks of extents δ may propagate.

The treatment of the problem is similar to that considered above, but requires some modifications. The magnitude of the forces P is now unknown and must be determined from knowledge of U. In the partially cracked system sketched in Figure 2 the displacement U is the sum of the displacement at the fracture tips $x = \pm(1-\delta)$ (evaluated in (3.14)) and that of the two pieces of detached stiffener at $x = \pm 1$:

$$U = \frac{2}{\pi\,E\,h}P(1-\delta)\left(\sum_{s=1,3,\ldots}^{2n+1}\frac{X_s}{s}\right) + \frac{P\delta}{E_c F_c}. \tag{4.1}$$

From this equation we derive P in terms of U and evaluate the strain energy $\frac{1}{2}PU$. Adding the surface energy we obtain the total energy E_{tot}:

$$E_{\text{tot}} = \frac{1}{2}2\frac{U^2}{\frac{2}{\pi\,E\,h}(1-\delta)\left(\sum_{x=1,3,\ldots}^{2n+1}\frac{X_s}{s}\right) + \frac{\delta}{E_c F_c}} + 2\gamma\delta. \tag{4.2}$$

Note that, in contrast to the case of impressed forces, the strain energy is now positive, since there are no external loads. Here again the analysis of the critical conditions is done for two extreme cases. If λ is neglegible with respect to 1 we can approximate the X_s in (4.1) by (3.16). After differentiation with respect to δ we arrive at the following expression for U_{cr}^2:

$$U_{\text{cr}}^2 = \frac{\frac{2\gamma}{E_c F_c}\left[\delta + \frac{2E_c F_c}{\pi\,E\,h}(1-\delta)\lambda\frac{\pi^2}{4}\right]}{1 - \frac{2E_c F_c}{E\,h}\lambda\frac{\pi}{4}}. \tag{4.3}$$

Also in this case U_{cr}^2 becomes theoretically infinite if we give λ the value $\frac{2Eh}{\pi E_c F_c}$. Another extreme situation occurs when $(1-\delta)\lambda$ is large, since now we may replace the approximate solution (3.18) in (4.2). Then an easy calculation of U_{cr}^2 yields the value

$$U_{\mathrm{cr}}^2 = \frac{2\gamma}{E_c F_c}.$$

It is useful to observe that, though being conceptually analogous as formulation and treatment, the two problems, of imposed forces or displacements, have strongly different solutions as far as the numerical and qualitative dependence of the critical values P_{cr}^2, U_{cr}^2 on δ is concerned.

5. Some concluding remarks

The problem of the brittle detachment of a stiffener from the edge of an elastic halfplane is, of course, only a schematic example of how to treat the mutual debonding of two elastic bodies as soon as the total energy of the combined system reaches a certain critical value. The critical state is characterized by the condition that the released elastic energy, due to the sudden creation of a fracture at the interface, is balanced by an equal supply of fracture energy localized along the surface of rupture. The mechanical criterion is simple, but the evaluation of the released elastic energy is difficult, even in the case of a single rectilinear stiffener bonded to a half-plane. The application of the energy theory of debonding are not limited to stiffeners in steel constructions. The criterion is now employed in order to estimate the resistance of materials reinforced by fibers, to predict the onset of earthquaques consequent to the formation of tectonic faults, to establish the duration of a prothesis and other interesting phenomena.

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